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NORBERT WIENER'S ERGODIC THEOREM FOR CONVEX REGIONS

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ABSTRACT. It is proved that the geometric hypothesis of a theorem which generalizes Norbert Wiener's multiparameter ergodic theorem are satsified in the case of arbitrary convex regions, provided only that they form a substantial family as defined in the introduction.

1. Introduction. Let T be the set of all points $t = (t_1, \ldots, t_k)$ with nonnegative coordinates in k-dimensional euclidean space and let $(\theta_t, t \in T)$ be a k-parameter semigroup of measure preserving transformations of the σ-finite measure space (X, μ). One of the authors has proved in [2] that if $(D_\alpha, \alpha > 0)$ is an increasing family of bounded regions in R^k whose union is T, then for every function f in $L^1(X)$, the averages

$$\frac{1}{|D_\alpha|} \int_{D_\alpha} f(\theta_t x)\, dt, \tag{1}$$

where vertical bars stand for Lebesgue measure, converge for almost all x as $\alpha \to \infty$, provided that the regions D_α satisfy the following hypothesis of geometric nature:

(A) There exists a family $(P_\alpha, \alpha > 0)$ of closed cells and a positive constant C such that for each α, $P_\alpha \supset D_\alpha$ and $|D_\alpha| \geqslant C|P_\alpha|$.

(B) For each t in R^k

$$\lim_{\alpha \to \infty} \frac{|(t + D_\alpha)\Delta D_\alpha|}{|D_\alpha|} = 0,$$

where Δ denotes the symmetric difference.

(C) If $B_{K,\alpha}$ is the set of all points t in R^k such that $t + D_\alpha$ intersects the compact set K without covering it, then

$$\lim_{\alpha \to \infty} \frac{|B_{K,\alpha}|}{|D_\alpha|} = 0.$$

Since the above stated conditions are satisfied when D_α is the intersection

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of T with the ball of radius α, that result was legitimately considered an extension of the now classical Wiener's ergodic theorem [3].

A family of regions D_α satisfying (A) is sometimes called a substantial family in differentiation theory; it has the purpose of granting the validity of the "maximal ergodic inequality" [2, Theorem 1].

As to the less familiar conditions (B) and (C), it is the aim of this note to prove that they are both satisfied if the regions D_α are convex.

2. **Convex regions.** According to the preceding, we shall assume that the sets of the family (D_α, $\alpha > 0$) are convex and compact regions whose union is T and that $\alpha < \beta$ implies $D_\alpha \subset D_\beta$. By "parallelotope" we mean in the sequel any set which is congruent to an interval in R^k. The parallelotope P is said to circumscribe a convex set C if the faces of P are contained in supporting hyperplanes of C.

Let K_α^u be the set of all points t in R^k whose distance to the complement of D_α is not less than the positive number u. Then K_α^u is convex, compact and contained in D_α, and condition (B) will be proved if we show that

$$\lim_{\alpha\to\infty} \frac{|K_\alpha^u|}{|D_\alpha|} = 1 \quad \text{for each positive number } u. \tag{2}$$

For if (2) is satisfied, given t in R^k, let us choose a number u which exceeds the length of the vector t. Then K_α^u is contained both in D_α and in $t + D_\alpha$. Denoting by $A \setminus B$ the set of points in A not in B, we have

$$\begin{aligned}\frac{|D_\alpha \Delta(t + D_\alpha)|}{|D_\alpha|} &= \frac{|D_\alpha \setminus (t + D_\alpha)| + |(t + D_\alpha) \setminus D_\alpha|}{|D_\alpha|} \\ &\leqslant \frac{|D_\alpha \setminus K_\alpha^u| + |(t + D_\alpha) \setminus K_\alpha^u|}{|D_\alpha|} = 2\left(1 - \frac{|K_\alpha^u|}{|D_\alpha|}\right),\end{aligned}$$

and the last expression tends to zero as $\alpha \to \infty$ by virtue of (2).

In order to prove (2) let us call P_α the parallelotope of minimum volume circumscribing D_α and let Q be the unit cube in R^k, defined by the relations $0 \leqslant t_i \leqslant 1$ ($i = 1, 2, \ldots, k$). We consider the affine transformation A_α such that $A_\alpha(P_\alpha) = Q$. Since $|P_\alpha| \to \infty$, it follows that $\det A_\alpha$ tends to zero as $\alpha \to \infty$. The sets $E_\alpha = A_\alpha(D_\alpha)$ and $L_\alpha^u = A_\alpha(K_\alpha^u)$ are convex, compact and contained in Q. Moreover

$$|E_\alpha| = |\det A_\alpha|\,|D_\alpha|, \qquad |L_\alpha^u| = |\det A_\alpha|\,|K_\alpha^u|.$$

On the other hand, we recall that the class of all nonvoid convex compact sets contained in Q is a metric space with respect to the distance Δ defined in [1, p. 60], and that this metric space is compact by virtue of Blaschke's selection theorem [1, p. 64], Therefore any sequence $\alpha_n \to \infty$ contains a subsequence α_{n_k} such that the limits

$$E = \lim_{k\to\infty} E_{\alpha_{n_k}}, \qquad L^u = \lim_{k\to\infty} L^u_{\alpha_{n_k}}$$

exist, are convex, compact, nonvoid and contained in Q. For simplicity of notation, we shall assume that $\alpha_n = n$ and that the subsequence coincides with the whole sequence.

Let now $\mathcal{C}$ be the class of all convex compact sets C in R^k such that Q is the parallelotope of minimum volume circumscribing C. We claim that $\mathcal{C}$ is a compact family in the above described metric space. For if C is the limit of a sequence C_n of sets in $\mathcal{C}$ and Q' is the parallelotope of minimum volume circumscribing C, we consider for each n the parallelotope Q_n which circumscribes C_n with the same orientation as Q' (i.e. with faces parallel to those of Q'). Then

$$Q' = \lim_{n\to\infty} Q_n, \quad \text{and } |Q_n| \geqslant |Q| \text{ for each } n.$$

Since volume is continuous, by letting $n \to \infty$ in the last relation, we obtain $|Q'| \geqslant |Q|$ so that, in fact, equality holds and C is a member of $\mathcal{C}$, which proves that $\mathcal{C}$ is closed, and therefore compact, as we claimed. Since $E_n \in \mathcal{C}$ for each n, it follows that $E \in \mathcal{C}$. This implies that the measure of E is positive (a convex set of measure zero cannot belong to $\mathcal{C}$, for it is contained in some hyperplane).

From the fact that the norm of the linear part of A_α, that we denote by $\|A_\alpha\|$, tends to zero as $\alpha \to \infty$ and from the inequality

$$\Delta(L^u_n, E_n) \leqslant \|A_n\| \cdot \Delta(K^u_n, D_n) \leqslant u \cdot \|A_n\|$$

we conclude that $L^u = E$. Finally, the obvious relations

$$\lim_{n\to\infty} \frac{|K^u_n|}{|D_n|} = \lim_{n\to\infty} \frac{|L^u_n|}{|E_n|} = \frac{|E|}{|E|} = 1$$

prove that (2), and therefore (B), hold in the case of convex regions.

In order to prove (C), we consider the set D^u_α consisting of all points t in R^k whose distance to D_α does not exceed the positive number u. The sets thus defined are convex, compact, and satisfy the relation

$$\lim_{\alpha\to\infty} \frac{|D^u_\alpha|}{|D_\alpha|} = 1 \quad \text{for each positive number } u, \tag{3}$$

whose proof we omit, since it is completely analogous to that of (2). If we take u equal to the diameter of the compact set K, then $t \in B_{K,\alpha}$ implies $(K - t) \subset D^u_\alpha$ and $(K - t) \cap K^u_\alpha = \varnothing$. Therefore

$$B_{K,\alpha} \subset \{t: (K - t) \subset D^u_\alpha \setminus K^u_\alpha\}, \qquad u = \operatorname{diam}(K),$$

from which it follows that $|B_{K,\alpha}| \leqslant |D^u_\alpha| - |K^u_\alpha|$ and also

$$\frac{|B_{K,\alpha}|}{|D_\alpha|} \leqslant \frac{|D^u_\alpha|}{|D_\alpha|} - \frac{|K^u_\alpha|}{|D_\alpha|}. \tag{4}$$

Since the right hand member of (4) tends to zero as $\alpha \to \infty$ by virtue of (2) and (3) our assertion is established.

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